characteristics is given in Weir's paper. Finally, the entropy distribution is given by $s = c_v \log y + \text{const.}$

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Final-Stage Decay of a Single Line Vortex

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I. Vortex Generation and Decay

The generation of a circulatory flow field by a vortex core of radius r_g was studied by Goldstein.[1] The differential system governing the fluid motion is

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)$$

$$t < 0: \quad v \equiv 0 \tag{1}$$

$$t > 0: \quad r = r_g, v = v_g; \; r = \infty, v = 0$$

and has the solution

$$v = \frac{v_g r_g}{r} + \frac{2v_g}{\pi}\int_0^\infty \exp(-\nu t x^2)\,\frac{J_1(xr)Y_1(xr_g) - Y_1(xr)J_1(xr_g)}{J_1^2(xr_g) + Y_1^2(xr_g)}\,\frac{dx}{x} \tag{2}$$

The limiting case of a vanishing core radius but finite centerline circulation Γ_g was shown by Rouse and Hsu[2] to reduce to following form:

$$v = (\Gamma_g/2\pi r)\exp(-r^2/4\nu t) \tag{3}$$

This corresponds to the vorticity diffusion from a line vortex of strength Γ_g, i.e.,[3]

$$\zeta = (\Gamma_g/4\pi\nu t)\exp(-r^2/4\nu t) \tag{4}$$

Thus the generation of a circulatory flow field by a vortex core is through vorticity diffusion, and, with finite time of generation, the velocity field so generated has limited extent. Moreover, such a velocity field is never and nowhere free of vorticity. Indeed, a potential flow never can be generated by a shear mechanism alone in finite time; it only represents a limit solution for $t \to \infty$. Thus, as $t \to \infty$, Eq. (3) describes a potential flow.

The spatial growth of velocity field and the time decay of kinetic energy of a viscous vortex after a generation period t_g can be studied simply by introducing, at $t = t_g$, an anti-circulation along the centerline. By simple superposition, one readily can write down the following result:

$$t = t_g + t_d$$

$$v = \frac{\Gamma_g}{2\pi r}\left[\exp\frac{-r^2}{4\nu(t_g + t_d)} - \exp\frac{-r^2}{4\nu t_d}\right] \tag{5}$$

This is indeed nothing else but the solution to the well-known Rayleigh "start-then-stop" problem for a circulatory flow field. The catching-up process between these two outward-propagating, oppositely orientated velocity fields results in a gradual annihilation of the momentum and a continuous decay of the kinetic energy of the flow field.

Equation (5) describes the velocity field of a viscous vortex at time t_d after its generation. It is clear that this flow field has not only limited spatial extent but also finite energy content.

Writing $A = \Gamma_g$, $a = 4\nu t_d$, $k = 1 + t_g/t_d$, Eq. (5) can be put in the form

$$v = (A/2\pi r)[\exp(-r^2/ak) - \exp(-r^2/a)] \tag{6}$$

It is easy to see that constants A, a, k can be used to define an initial vortex field, provided that k is not taken too close to unity. The parameter k indeed fixes the age of the initial vortex field; for $t_d/t_g \to 0$, $k \to \infty$, and for $t_d/t_g \to \infty$, $k \to 1$.

A, a, k can be related further to the measurable field characteristics r_0, Γ_0, ζ_0 as follows: defining r_0 as the initial radius of maximum circulation Γ_0 ($\Gamma = 2\pi r v$), i.e., for

$$\frac{\partial\Gamma}{\partial r} = \frac{2Ar}{a}\left(\exp\frac{-r^2}{a} - \frac{1}{k}\exp\frac{-r^2}{ka}\right) = 0$$

one has

$$r_0 = \{[ka/(k-1)]\ln k\}^{1/2} \tag{7}$$

$$\Gamma_0 = A(k-1)k^{k/(1-k)} \tag{8}$$

Now, vorticity is given by

$$\zeta = \frac{1}{r}\frac{\partial}{\partial r}(vr) = \frac{A}{\pi k a}\left(k\exp\frac{-r^2}{a} - \exp\frac{-r^2}{ka}\right)$$

so, putting $r = 0$, one has for the centerline vorticity

$$\zeta_0 = (A/\pi k a)(k-1) \tag{9}$$

For a particular set of A, a, k (or initial values r_0,ζ_0,Γ_0), the subsequent velocity field now can be put in the form

$$v = \frac{A}{2\pi r}\left(\exp\frac{-r^2}{ak + 4\nu t} - \exp\frac{-r^2}{a + 4\nu t}\right) \tag{10}$$

The vortex-core radius r_c ($r = r_c$, $\partial\Gamma/\partial r = 0$) readily can be obtained as follows:[2]

$$r_c^2 = a\frac{(k + 4\nu t/a)(1 + 4\nu t/a)}{k-1}\ln\frac{k + 4\nu t/a}{1 + 4\nu t/a} \tag{11}$$

The kinetic energy per unit length of vortex is given by an integral:

$$E = \pi\rho\int_0^\infty v^2 r\,dr = \frac{\rho A^2}{8\pi}\int_0^\infty \frac{1}{r^2}\left(\exp\frac{-r^2}{ka + 4\nu t} - \exp\frac{-r^2}{a + 4\nu t}\right)^2 2r\,dr$$

Upon introducing the identity

$$\int_0^\infty [\exp(-px) - \exp(-qx)]^2\frac{dx}{x} = \ln\frac{p+q}{2p} + \ln\frac{p+q}{2q}$$

one obtains[2]

$$E = \frac{\rho A^2}{8\pi}\ln\frac{[1 + (k + 4\nu t/a)/(1 + 4\nu t/a)]^2}{4(k + 4\nu t/a)/(1 + 4\nu t/a)} \tag{12}$$

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II. Final-Stage Decay Law

When $4\nu t/a = t/t_d \gg 1$, and k is not close to 1 [i.e., $t_d/t_g \sim 0(1)$], asymptotic expansions of Eqs. (11) and (12) for large t lead to the following result. Putting $a/4\nu t = \beta$, $ka/4\nu t = \alpha$, and $\alpha = k\beta$, one obtains, from Eq. (11)

$$r_c^2 = \frac{a}{\beta}\frac{(1+\alpha)(1+\beta)}{(\alpha-\beta)}\ln\left(\frac{1+\alpha}{1+\beta}\right) = \frac{a}{\beta}\left[1 + (k+1)\frac{\beta}{2} + \ldots\right]$$

i.e.,

$$r_c = (4\nu t)^{1/2} \tag{13}$$

From Eq. (12), one obtains

$$E = \frac{\rho A^2}{8\pi}\left[2\ln\left(1 + \frac{1+\alpha}{1+\beta}\right) - \ln\left(\frac{1+\alpha}{1+\beta}\right) - \ln 4\right]$$

i.e.,

$$E = \frac{\rho A^2}{32\pi}(k-1)^2\left(\frac{a}{4\nu t}\right)^2 \tag{14}$$

Expressions (13) and (14) constitute the two asymptotic laws that govern the final-stage vortex-core growth and kinetic-energy decay, i.e., "the radius of the vortex core grows in proportion to the square root of the decay time; the content of kinetic energy decays according to the inverse square of the decay time."

It might be of interest to note that the vortex-core radius r_c in Eq. (13) coincides with Taylor's dissipation eddy scale λ of a homogenous turbulence field, although they certainly follow different decay laws. That λ must correspond to a $(-\frac{5}{2})$-power energy-decay law is evident from the dissipation equation

$$\partial u'^2/\partial t = -10\nu u'^2/\lambda^2 \tag{15}$$

The significance of the coincidence, however, will be discussed in a future report.

The inverse-square law for final-stage decay of a viscous vortex [Eq. (14)] follows strictly from the limited time of generation and finite amount of field-energy content. It is not difficult to show that this decay law differs essentially from the decay law corresponding to Lamb's model of viscous decay of a potential vortex field. Indeed, for Lamb's model, the following result was obtained:[3]

$$v = (\Gamma_0/2\pi r)[1 - \exp(-r^2/4\nu t)] \tag{16}$$

Comparing Eqs. (16) and (5), it is at once clear that Lamb's model of vortex decay implies infinite generation time and infinite flow-field kinetic energy.

It might not be out of place at this point to remark that the present single-vortex final-stage energy-decay law, i.e., Eq. (14), played an essential role in the formulation of a recent theory on the final-stage decay of grid-produced turbulence.[4,5]

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Free Vibration of Rectangular and Circular Orthotropic Plates

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Nomenclature

A_1,A_2,A_3 A_4,A_5	= constants
a,b	= dimensions of plate
D_x,D_y	= flexural rigidities of orthotropic plate
c	= radius of circular plate
D	= flexural rigidity of isotropic plate
H	= torsional rigidity of orthotropic plate
h	= plate thickness
k	= damping coefficient
r,θ	= polar coordinates
t	= time
$w(x,y,t)$	= deflection of plate
x,y	= rectangular coordinates
ω	= circular frequency
ρ	= mass density of plate material
ν	= Poisson's ratio
$\nabla_0{}^4$	= $(D_x\partial^4/\partial x^4) + (2H\partial^4/\partial x^2\partial y^2) + (D_y\partial^4/\partial y^4)$
$\nabla_{11}{}^4\psi$	= $D_x\psi_{,rrrr} + 2H[(2/r^3)\psi_{,r} - (2/r^2)\psi_{,rr} + (1/r)\psi_{,rrr}] + D_y[-(3/r^3)\psi_{,r} + (3/r^2)\psi_{,rr}]$

Subscripts

t,tt,r,rr, etc. = derivatives with respect to t and r, respectively

THE frequencies of the fundamental normal modes of free vibration for rectangular and circular orthotropic plates are obtained by using the Galerkin method as formulated by Stanišić.[1] The results are compared with values obtained by other methods. Although vibration analysis of rectangular orthotropic plates has been considered by many writers,[2, 3] the case of circular orthotropic places has not received much attention.

Method of Solution

Assuming the classical small-deflection theory, neglecting rotatory inertia, and taking the damping forces to be proportional to the velocity, the motion of the orthotropic plate is governed by the following differential equation:

$$\nabla_0{}^4 w + kw_{,t} + \rho h w_{,tt} = 0 \tag{1}$$

The solution for w can be taken as

$$w(x,y,t) = f(x,y)\phi(t) \tag{2}$$

where $f(x,y)$ is the characteristic function chosen to satisfy the boundary conditions. So

$$\delta w = f(x,y)\delta\phi(t) \tag{3}$$

For a virtual displacement δw, the criterion according to Galerkin can be stated as

$$\iint L(w)\delta w\, dx\, dy = 0 \tag{4}$$

where $L(w)$ stands for the expression on left side of Eq. (1), and the double integral is taken over the area of the plate.

Taking

$$\phi(t) = e^{-\alpha t}\cos\omega t \tag{5}$$

and substituting Eqs. (2) and (3) in Eq. (1) gives, for all t,

$$\nabla_0{}^4 f(x,y) - \lambda^2 f(x,y) = 0 \tag{6}$$

where λ^2 is given by

$$\omega^2 = (\lambda^2/\rho h) - (k/2\rho h)^2 \tag{7}$$

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